Alignment and Matching Tests for High-Dimensional Tensor Signals

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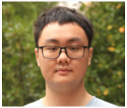


School of Data Science
The Chinese University of Hong Kong, Shenzhen (CUHKSZ)

2025 Workshop on Statistical Network Analysis and Beyond, Tokyo

Introduction

Tensor models

- Kolda and Bader [2009] demonstrated that the high-order tensor data often exhibit intrinsic low-rank structures, which are key to efficient analysis.
- To capture the low-rank structures of the high-order tensor, we consider the following d-fold rank-R spiked tensor model:

$$\boldsymbol{T} = \sum_{r=1}^{R} \beta_r \boldsymbol{x}^{(r,1)} \otimes \cdots \otimes \boldsymbol{x}^{(r,d)} + \frac{1}{\sqrt{N}} \boldsymbol{X}. \tag{1}$$

- Key components:
  - $\beta_1 \geq \cdots \geq \beta_R > 0$ are the signal-to-noise ratios (SNRs), indicating the strength of the signal.
  - $\{\boldsymbol{x}^{(1,l)}, \cdots, \boldsymbol{x}^{(R,l)}\}$ are mutually orthogonal unit vectors in $\mathbb{R}^{n_l}$ for $1 \leq l \leq d$.
  - $\boldsymbol{X} = [X_{i_1 \cdots i_d}] \in \mathbb{R}^{n_1 \times \cdots \times n_d}$ is a noise tensor with i.i.d. entries $X_{i_1 \cdots i_d}$ such that $\mathbb{E}[X_{i_1 \cdots i_d}] = 0$ and $\mathbb{E}[X^2_{i_1 \cdots i_d}] = 1$.
  - The tensor dimensions $n_1, \cdots, n_d$ tend to infinity proportionally.
  - $N = n_1 + \cdots + n_d$.

Our goal

- Majority of literature focuses on "recovering signal vectors" from the observed tensor $\boldsymbol{T}$.
- Difficulty: Challenges arise when SNRs fall below a critical threshold (phase transition), making recovery extremely difficult or even theoretically impossible.
- Our goal: hypothesis testing: We consider the following two hypothesis testing problems:
    - Tensor Signal Alignment: Assessing whether two tensor signals can be aligned.
    - Tensor Signal Matching: Determining if two tensor signals are statistically equivalent.

Test 1: tensor signal alignment

Let $\boldsymbol{T}$ be a d-dimensional tensor defined as

$$\boldsymbol{T} = \sum_{r=1}^{R} \beta_r \boldsymbol{x}^{(r,1)} \otimes \cdots \otimes \boldsymbol{x}^{(r,d)} + \frac{1}{\sqrt{N}} \boldsymbol{X},$$

test the alignment of tensor signals with a given directional tensor $\boldsymbol{a}^{(1)} \otimes \cdots \otimes \boldsymbol{a}^{(d)}$, where $\boldsymbol{a}^{(l)} \in \mathbb{R}^{n_l}$ are deterministic unit vectors for $1 \leq l \leq d$:

$H_0 : \boldsymbol{a}^{(l)} \perp \boldsymbol{x}^{(r,l)} \quad \text{for } 1 \leq l \leq d, 1 \leq r \leq R,$

H_1 : There exists at least one $1 \leq l \leq d, 1 \leq r \leq R$ such that $\boldsymbol{a}^{(l)} \not\perp \boldsymbol{x}^{(r,l)}$.

Test 2: tensor signal matching

Consider two independent tensor observations $\boldsymbol{T}^{(0)}$ and $\boldsymbol{T}^{(1)}$ defined as:

$$\begin{cases} \boldsymbol{T}^{(0)} = \sum_{r_0=1}^{R_0} \beta_{r_0,0} \boldsymbol{x}^{(r_0,1)} \otimes \cdots \otimes \boldsymbol{x}^{(r_0,d)} + \frac{1}{\sqrt{N}} \boldsymbol{X}^{(0)}, \\ \boldsymbol{T}^{(1)} = \sum_{r_1=1}^{R_1} \beta_{r_1,1} \boldsymbol{y}^{(r_1,1)} \otimes \cdots \otimes \boldsymbol{y}^{(r_1,d)} + \frac{1}{\sqrt{N}} \boldsymbol{X}^{(1)}, \end{cases}$$

where $\boldsymbol{X}^{(0)}$ and $\boldsymbol{X}^{(1)}$ are independent and $\boldsymbol{x}^{(r_0,l)}, \boldsymbol{y}^{(r_1,l)} \in \mathbb{R}^{n_l}$ are deterministic unit vectors for $1 \le l \le d$, $1 \le r_0 \le R_0$, $1 \le r_1 \le R_1$. Test the tensor signal matching can be formulated as the following hypothesis test:

$$\begin{aligned} H_0 :& \boldsymbol{x}^{(r_0,l)} \perp \boldsymbol{y}^{(r_1,l)} \text{ for any } 1 \le r_0 \le R_0, 1 \le r_1 \le R_1 \text{ and } 1 \le l \le d, \\ H_1 :& \text{There exists at least one } 1 \le r_0 \le R_0, 1 \le r_1 \le R_1 \text{ and } 1 \le l \le d \\ & \text{such that } \boldsymbol{x}^{(r_0,l)} \not\perp \boldsymbol{y}^{(r_1,l)}. \end{aligned}$$

Tensor contraction operator

The *tensor contraction operator* $\boldsymbol{\Phi}_d$ (introduced in Seddik et al. [2024]) maps a d-fold tensor $\boldsymbol{T} \in \mathbb{R}^{n_1 \times \cdots \times n_d}$ and unit vectors $\boldsymbol{a}^{(1)} \in \mathbb{R}^{n_1}, \cdots, \boldsymbol{a}^{(d)} \in \mathbb{R}^{n_d}$ to a matrix representation:

$$\boldsymbol{R} = \boldsymbol{\Phi}_d(\boldsymbol{T}, \boldsymbol{a}^{(1)}, \cdots, \boldsymbol{a}^{(d)}) = \begin{pmatrix} \mathbf{0}_{n_1 \times n_1} & \boldsymbol{T}^{12} & \cdots & \boldsymbol{T}^{1d} \\ (\boldsymbol{T}^{12})' & \mathbf{0}_{n_2 \times n_2} & \cdots & \boldsymbol{T}^{2d} \\ \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{T}^{1d})' & (\boldsymbol{T}^{2d})' & \cdots & \mathbf{0}_{n_d \times n_d} \end{pmatrix}, \tag{2}$$

where for $1 \leq k < k' \leq d$,

$$\boldsymbol{T}^{kk'} = \left[\sum_{i_j = 1, j \neq k, k'}^{n_j} T_{i_1 \cdots i_d} \prod_{\ell = 1, \ell \neq k, k'}^{d} a_{i_\ell}^{(\ell)} \right]_{n_k \times n_{k'}} \tag{3}$$

is an $n_k \times n_{k'}$ matrix, called *second order contraction matrix* of $\boldsymbol{T}$ along the directions $\{\boldsymbol{a}^{(k)}, \boldsymbol{a}^{(k')}\}$, introduced in Lim [2005].

Example

For example, when $d = 4$, we have

$$\boldsymbol{\Phi}_d(\boldsymbol{T}, \boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}, \boldsymbol{a}^{(3)}, \boldsymbol{a}^{(4)}) = \begin{pmatrix} \mathbf{0}_{n_1 \times n_1} & \boldsymbol{T}^{12} & \boldsymbol{T}^{13} & \boldsymbol{T}^{14} \\ * & \mathbf{0}_{n_2 \times n_2} & \boldsymbol{T}^{23} & \boldsymbol{T}^{23} \\ * & * & \mathbf{0}_{n_3 \times n_3} & \boldsymbol{T}^{34} \\ * & * & * & \mathbf{0}_{n_4 \times n_4} \end{pmatrix},$$

where $*$ denotes the symmetry, and

$$\boldsymbol{T}^{12} = \left[\sum_{i_3=1}^{n_3} \sum_{i_4=1}^{n_4} T_{i_1 i_2 i_3 i_4} a_{i_3}^{(3)} a_{i_4}^{(4)} \right]_{n_1 \times n_2},$$

$$\boldsymbol{T}^{13} = \left[\sum_{i_2=1}^{n_2} \sum_{i_4=1}^{n_4} T_{i_1 i_2 i_3 i_4} a_{i_2}^{(2)} a_{i_4}^{(4)} \right]_{n_1 \times n_3},$$

$$\boldsymbol{T}^{14} = \left[\sum_{i_2=1}^{n_2} \sum_{i_3=1}^{n_3} T_{i_1 i_2 i_3 i_4} a_{i_2}^{(2)} a_{i_3}^{(3)} \right]_{n_1 \times n_4},$$

etc.

Basic properties of the contraction operator $\mathbf{\Phi}_d$

- For the d-fold rank-R spiked tensor $\boldsymbol{T}$, since $\mathbf{\Phi}_d$ is linear in $\boldsymbol{T}$, we have

$$\boldsymbol{R} = \mathbf{\Phi}_d(\boldsymbol{T}, \boldsymbol{a}^{(1)}, \cdots, \boldsymbol{a}^{(d)}) = \sum_{r=1}^{R} \beta_r \mathbf{\Phi}_d(\boldsymbol{x}^{(r,1)} \otimes \cdots \otimes \boldsymbol{x}^{(r,d)}, \boldsymbol{a}^{(1)}, \cdots, \boldsymbol{a}^{(d)})$$
$$+ \frac{1}{\sqrt{N}} \mathbf{\Phi}_d(\boldsymbol{X}, \boldsymbol{a}^{(1)}, \cdots, \boldsymbol{a}^{(d)}) = \boldsymbol{S} + \boldsymbol{M}.$$

- For the tensor signal alignment test, under the null hypothesis H_0, i.e. $\boldsymbol{a}^{(l)} \perp \boldsymbol{x}^{(r,l)}$ for $1 \leq l \leq d$ and $1 \leq r \leq R$, we have $\boldsymbol{S} = \mathbf{0}$, so $\boldsymbol{R} = \boldsymbol{M}$.
- In contrast, under the alternative H_1 of the tensor signal alignment test, $\boldsymbol{S} \neq \mathbf{0}$, result in $\boldsymbol{R} \neq \boldsymbol{M}$. This difference is the key to distinguish H_0 and H_1.

Test statistic for the alignment test

- The statistic of the tensor signal alignment test is defined as:

$$\widehat{T}_N^{(d)} = \|\boldsymbol{R}\|_2^2 - N \int_{-\infty}^{\infty} x^2 \nu(dx), \tag{4}$$

where ν is the limiting spectral distribution (LSD) of $\boldsymbol{R}$.

- Under the null hypothesis H_0:

$$\left(\widehat{T}_N^{(d)} - \xi_N^{(d)}\right) / \sigma_N^{(d)} \xrightarrow{d} \mathcal{N}(0, 1).$$

- Under the alternative hypothesis H_1:

$$\left(\widehat{T}_N^{(d)} - \xi_N^{(d)} - \mathcal{D}^{(d)}\right) / \sigma_N^{(d)} \xrightarrow{d} \mathcal{N}(0, 1),$$

where $\mathcal{D}^{(d)}$ represents a positive mean drift, indicating an effect size.

Main contributions

- We conduct an in-depth analysis of the contracted data matrix $\boldsymbol{R}$, whose entries display significant correlations and deviate from traditional random matrix models in which the elements of the noise matrix are typically assumed to be independent of one another, including

  (a) The characterization of its LSD through a vector Dyson equation, along with entrywise behaviors of the resolvent.

  (b) The establishment of CLT for a broad class of its LSS.

- We establish a rigorous procedure for the tensor signal alignment test by establishing the normality asymptotic of the test statistic and deriving its power function under a general alternative hypothesis.

- We also address the problem of testing for the matching of two high-dimensional low-rank tensor signals. To tackle this problem, we employ an approach similar to the one established for the tensor signal alignment test.

Assumptions

Assumption (Subexponential tails)

The noise variables $X_{i_1 \dots i_d}$ are i.i.d. with zero mean, unit variance and subexponential tails, that is, for some $\theta > 0$,

$$\limsup_{x \to \infty} e^{x^\theta} \mathbb{P}(|X_{i_1 \dots i_d}| \geq x) < \infty.$$

Assumption (High-dimensionality scheme)

The tensor dimensions $n_1, \cdots, n_d$ all tend to infinity in such a way that

$$\lim_{n_1, \cdots, n_d \to \infty} \frac{n_j}{n_1 + \cdots + n_d} = \mathfrak{c}_j \in (0, 1), \quad 1 \leq j \leq d.$$

We define $N := n_1 + \cdots + n_d \to \infty$ and

$$\mathfrak{c} = (\mathfrak{c}_1, \cdots, \mathfrak{c}_d)'. \tag{5}$$

The limiting spectral distribution of the matrix M

Structures of the matrix $\boldsymbol{M}$

- Recall the definition of $\boldsymbol{M}$ is as follows:

$$\boldsymbol{M} = \frac{1}{\sqrt{N}} \boldsymbol{\Phi}_d(\boldsymbol{X}, \boldsymbol{a}^{(1)}, \cdots, \boldsymbol{a}^{(d)}) = \frac{1}{\sqrt{N}} \begin{pmatrix} \mathbf{0}_{n_1 \times n_1} & \boldsymbol{X}^{12} & \cdots & \boldsymbol{X}^{1d} \\ (\boldsymbol{X}^{12})' & \mathbf{0}_{n_2 \times n_2} & \cdots & \boldsymbol{X}^{2d} \\ \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{X}^{1d})' & (\boldsymbol{X}^{2d})' & \cdots & \mathbf{0}_{n_d \times n_d} \end{pmatrix},$$

  where $\boldsymbol{X}^{l_1 l_2} \in \mathbb{R}^{n_{l_1} \times n_{l_2}}$ is constructed as in (3).

- The entries of $\boldsymbol{M}$ are generally correlated.

$$d = 3: \quad \mathrm{Cov}(X^{l_1 l_2}_{s_1, t_1}, X^{l_1 l_3}_{s_2, t_2}) = \delta_{s_1, s_2} a^{(l_2)}_{t_1} a^{(l_3)}_{t_2},$$
$$d \geq 4: \quad \mathrm{Cov}(X^{i_1 j_1}_{s_1, t_1}, X^{i_2 j_2}_{s_2, t_2}) = a^{(i_1)}_{s_1} a^{(i_2)}_{s_2} a^{(j_1)}_{t_1} a^{(j_2)}_{t_2}.$$

- While $\boldsymbol{M}$ is symmetric and its entries have zero mean and a variance of N^{-1}, similar to a standard Wigner matrix, these widespread correlations among $\boldsymbol{X}$'s entries complicate the theoretical analysis of the LSD of $\boldsymbol{M}$.

Vector Dyson equation and the LSD ν of $\boldsymbol{M}$

- To study the LSD of $\boldsymbol{M}$, we start by examining its resolvent matrix:

$$\boldsymbol{Q}(z) := (\boldsymbol{M} - z\boldsymbol{I}_N)^{-1}, \quad z \in \mathbb{C}^+. \tag{6}$$

- vector Dyson equation induced by the matrix $\boldsymbol{M}$:

$$-\frac{\mathfrak{c}}{\boldsymbol{g}(z)} = z + \boldsymbol{S}_d \boldsymbol{g}(z), \tag{7}$$

  where

$$\boldsymbol{g}(z) = (g_1(z), \cdots, g_d(z))',$$
$$\boldsymbol{S}_d = \mathbf{1}_{d \times d} - \boldsymbol{I}_d.$$

- The solution $\boldsymbol{g}(z)$ is unique, and $m(z) = g_1(z) + \cdots + g_d(z)$ is the Stieltjes transform of the LSD ν of $\boldsymbol{M}$.

CLT for LSS

CLT for the LSS of the matrix $\boldsymbol{M}$

- Recall that our statistic $\widehat{T}_N^{(d)}$ in (4) for the tensor signal alignment test is an LSS of the matrix $\boldsymbol{M}$ under H_0.
- We will establish the CLT for a broad class of the LSS of the matrix $\boldsymbol{M}$.
- For simplicity, we only present the CLT for 3-fold tensors, as the general case of $d \geq 3$ involve more complicated formulas, though without fundamental difference.

LSS of the matrix $\boldsymbol{M}$

- Define $v_B^{(3)} := \max\{\zeta, \mathfrak{v}_3\}$ and consider the class of functions

$$\mathfrak{F}_3 := \left\{ f(z) : f \text{ is analytic on an open set containing } \left[-v_B^{(3)}, v_B^{(3)} \right] \right\}. \tag{8}$$

- For $f \in \mathfrak{F}_3$, consider a LSS of $\boldsymbol{M}$ of the form:

$$\mathcal{L}_{\boldsymbol{M}}(f) := \frac{1}{N} \sum_{l=1}^{N} f(\lambda_l) = \int_{\mathbb{R}} f(x) \nu_N(dx), \tag{9}$$

where $\lambda_1, \cdots, \lambda_N$ are the eigenvalues of $\boldsymbol{M}$ and $\nu_N = N^{-1} \sum_{j=1}^{N} \delta_{\lambda_j}$ is the ESD of $\boldsymbol{M}$.

- We will derive the asymptotic distribution of $G_N(f)$ as follows:

$$G_N(f) := N \int_{-\infty}^{\infty} f(x)(\nu_N(dx) - \nu(dx)) = N\left(\mathcal{L}_{\boldsymbol{M}}(f) - \int_{-\infty}^{\infty} f(x)\nu(dx) \right),$$

where ν is the LSD of $\boldsymbol{M}$.

Main result

Theorem

Under Assumptions 1.1 and 1.2 with $d = 3$, *for any* $f \in \mathfrak{F}_3$ *in* (8) *and deterministic unit vectors* $\boldsymbol{a}^{(1)} \in \mathbb{R}^{n_1}, \boldsymbol{a}^{(2)} \in \mathbb{R}^{n_2}, \boldsymbol{a}^{(3)} \in \mathbb{R}^{n_3}$, *we have*

$$\left(G_N(f) - \xi_N^{(3)}\right)/\sigma_N^{(3)} \xrightarrow{d} \mathcal{N}(0,1),$$

where

$$\xi_N^{(3)} := -\frac{1}{2\pi \mathrm{i}} \oint_{\mathfrak{C}_1} f(z)\mu_N^{(3)}(z; \kappa_3, \kappa_4, \boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}, \boldsymbol{a}^{(3)})dz, \tag{10}$$

$$(\sigma_N^{(3)})^2 := -\frac{1}{4\pi^2} \oint_{\mathfrak{C}_1} \oint_{\mathfrak{C}_2} f(z_1)f(z_2)\mathcal{C}_N^{(3)}(z_1, z_2; \kappa_4, \boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}, \boldsymbol{a}^{(3)})dz_1 dz_2, \tag{11}$$

and $\mathfrak{C}_1, \mathfrak{C}_2$ *are two disjoint rectangular contours with vertices* $\pm E_1 \pm \mathrm{i}\eta_1$ *and* $\pm E_2 \pm \mathrm{i}\eta_2$, *respectively, such that* $E_1, E_2 \geq v_B^{(3)} + t$, *where* $t > 0$ *is a fixed constant and* $\eta_1, \eta_2 > 0$.

Hypothesis tests

Test 1: tensor signal alignments

- **3-fold rank-R spiked tensor model:**

$$\boldsymbol{T} = \sum_{r=1}^{R} \beta_r \boldsymbol{x}^{(r,1)} \otimes \boldsymbol{x}^{(r,2)} \otimes \boldsymbol{x}^{(r,3)} + \frac{1}{\sqrt{N}} \boldsymbol{X}.$$

- **Tensor signal alignments test:** For three deterministic unit vectors $\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}, \boldsymbol{a}^{(3)}$, consider

$$H_0 : \boldsymbol{a}^{(l)} \perp \boldsymbol{x}^{(r,l)} \quad \text{for } 1 \leq l \leq 3, 1 \leq r \leq R,$$

$$H_1 : \text{There exists at least one } 1 \leq l \leq 3, 1 \leq r \leq R \text{ such that } \boldsymbol{a}^{(l)} \not\perp \boldsymbol{x}^{(r,l)}.$$

- **Test statistic:**

$$\widehat{T}_N^{(3)} = \|\boldsymbol{R}\|_2^2 - N \int_{-\infty}^{\infty} x^2 \nu(dx),$$

where $\boldsymbol{R} = \boldsymbol{\Phi}_3(\boldsymbol{T}, \boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}, \boldsymbol{a}^{(3)})$.

Proposition

Under Assumptions 1.1 and 1.2, the statistic $\widehat{T}_N^{(3)}$ satisfies that

$$\left(\widehat{T}_N^{(3)} - \xi_N^{(3)} - \mathcal{D}^{(3)}\right)/\sigma_N^{(3)} \xrightarrow{d} \mathcal{N}(0,1),$$

where

$$\mathcal{D}^{(3)} := 2\sum_{r=1}^{R} \beta_r^2 \sum_{l=1}^{3} \langle \boldsymbol{x}^{(r,l)}, \boldsymbol{a}^{(l)} \rangle^2 \geq 0, \tag{12}$$

and $\xi_N^{(3)}, \sigma_N^{(3)}$ are derived from (10) and (11) by setting $f(z) = z^2$,

$$\xi_N^{(3)} = -\frac{1}{2\pi \mathrm{i}} \oint_{\mathfrak{C}_1} z^2 \mu_N^3(z; \kappa_3, \kappa_4, \boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}, \boldsymbol{a}^{(3)}) dz,$$

$$(\sigma_N^{(3)})^2 = -\frac{1}{4\pi^2} \oint_{\mathfrak{C}_1} \oint_{\mathfrak{C}_2} z_1^2 z_2^2 \mathcal{C}_N^{(3)}(z_1, z_2; \kappa_4, \boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}, \boldsymbol{a}^{(3)}) dz_1 dz_2.$$

Rejection region and power

- We conclude from Proposition 4.1 that

$$\begin{cases} \left(\widehat{T}_N^{(3)} - \xi_N^{(3)}\right)/\sigma_N^{(3)} \xrightarrow{d} \mathcal{N}(0,1) & \text{under } H_0, \\ \left(\widehat{T}_N^{(3)} - \xi_N^{(3)} - \mathcal{D}^{(3)}\right)/\sigma_N^{(3)} \xrightarrow{d} \mathcal{N}(0,1) & \text{under } H_1. \end{cases} \tag{13}$$

- Given a significance level $\alpha \in (0,1)$, the rejection region of our test procedure is

$$\left\{\text{Reject } H_0 \text{ if } \left(\widehat{T}_N^{(3)} - \xi_N^{(3)}\right)/\sigma_N^{(3)} > z_\alpha\right\}, \tag{14}$$

where z_α is α-th upper quantile of the standard normal.

- The asymptotic power of our test satisfies that

$$\lim_{N\to\infty} \mathbb{P}(\left(\widehat{T}_N^{(3)} - \xi_N^{(3)}\right)/\sigma_N^{(3)} > z_\alpha | H_1) - 1 + \Phi(z_\alpha - \mathcal{D}^{(3)}/\sigma_N^{(3)}) = 0,$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal.

Test 2: tensor signal matching

- Consider two independent tensors $\boldsymbol{T}^{(0)}$ and $\boldsymbol{T}^{(1)}$ as follows:

$$\begin{cases} \boldsymbol{T}^{(0)} = \sum_{r_0=1}^{R_0} \beta_{r_0,0} \boldsymbol{x}^{(r_0,1)} \otimes \boldsymbol{x}^{(r_0,2)} \otimes \boldsymbol{x}^{(r_0,3)} + \frac{1}{\sqrt{N}} \boldsymbol{X}^{(0)}, \\ \boldsymbol{T}^{(1)} = \sum_{r_1=1}^{R_1} \beta_{r_1,1} \boldsymbol{y}^{(r_1,1)} \otimes \boldsymbol{y}^{(r_1,2)} \otimes \boldsymbol{y}^{(r_1,3)} + \frac{1}{\sqrt{N}} \boldsymbol{X}^{(1)}, \end{cases}$$

  where $\boldsymbol{X}^{(0)}$ and $\boldsymbol{X}^{(1)}$ are independent and $\boldsymbol{x}^{(r_0,l)}, \boldsymbol{y}^{(r_1,l)} \in \mathbb{R}^{n_l}$ are deterministic unit vectors for $1 \le l \le 3$ and $1 \le r_0 \le R_0, 1 \le r_1 \le R_1$.

- **Tensor signal matching test:**

$$\begin{aligned} &H_0 : \boldsymbol{x}^{(r_0,l)} \perp \boldsymbol{y}^{(r_1,l)} \text{ for any } 1 \le r_0 \le R_0, 1 \le r_1 \le R_1 \text{ and } 1 \le l \le 3, \\ &H_1 : \text{there } \exists \text{ at least one } 1 \le r_0 \le R_0, 1 \le r_1 \le R_1 \text{ and } 1 \le l \le 3 \text{ s.t. } \boldsymbol{x}^{(r_0,l)} \not\perp \boldsymbol{y}^{(r_1,l)}. \end{aligned}$$

- **Test statistic:** define $\boldsymbol{R}^{(r_0,1)} := \boldsymbol{\Phi}_d(\boldsymbol{T}^{(1)}, \boldsymbol{x}^{(r_0,1)}, \boldsymbol{x}^{(r_0,2)}, \boldsymbol{x}^{(r_0,3)})$ for $1 \le r_0 \le R_0$ and

$$\widehat{T}^{(3)}_{r_0,N} = \widehat{T}^{(3)}_{r_0,N}(\boldsymbol{x}^{(r_0,1)}, \boldsymbol{x}^{(r_0,2)}, \boldsymbol{x}^{(r_0,3)}) := \|\boldsymbol{R}^{(r_0,1)}\|_2^2 - N \int_{-\infty}^{\infty} x^2 \nu(dx). \tag{15}$$

CLT for the $\widehat{T}^{(3)}_{r_0,N}$

Proposition

Under Assumptions 1.1 and 1.2, the statistic $\widehat{T}^{(3)}_{r_0,N}$ satisfies that

$$\left(\widehat{T}^{(3)}_{r_0,N} - \xi_N^{(r_0,3)} - \mathcal{D}^{(r_0,3)}\right)/\sigma_N^{(r_0,3)} \xrightarrow{d} \mathcal{N}(0,1),$$

where

$$\mathcal{D}^{(r_0,3)} := 2\sum_{r_1=1}^{R_1} \beta_{r_1,1}^2 \sum_{l=1}^{3} \langle \boldsymbol{x}^{(r_0,l)}, \boldsymbol{y}^{(r_1,l)} \rangle^2 \geq 0,$$

and $\xi_N^{(r_0,3)}, \sigma_N^{(r_0,3)}$ are derived from (10) and (11) by setting $f(z) = z^2$, i.e.

$$\xi_N^{(r_0,3)} = -\frac{1}{2\pi \mathrm{i}} \oint_{\mathfrak{C}_1} z^2 \mu_N^{(3)}(z; \kappa_3, \kappa_4, \boldsymbol{x}^{(r_0,1)}, \boldsymbol{x}^{(r_0,2)}, \boldsymbol{x}^{(r_0,3)}) dz,$$

$$(\sigma_N^{(r_0,3)})^2 = -\frac{1}{4\pi^2} \oint_{\mathfrak{C}_1} \oint_{\mathfrak{C}_2} z_1^2 z_2^2 \mathcal{C}_N^{(3)}(z_1, z_2; \kappa_4, \boldsymbol{x}^{(r_0,1)}, \boldsymbol{x}^{(r_0,2)}, \boldsymbol{x}^{(r_0,3)}) dz_1 dz_2.$$

Numerical experiments

Experiment 1: test for signal alignment

- This experiment focuses on the tensor signal alignment test. We generate the observation $\boldsymbol{T}$ as follows with varying values of β.

$$\boldsymbol{T} = \beta \boldsymbol{x}^{(1)} \otimes \boldsymbol{x}^{(2)} \otimes \boldsymbol{x}^{(3)} + \frac{1}{\sqrt{N}} \boldsymbol{X}.$$

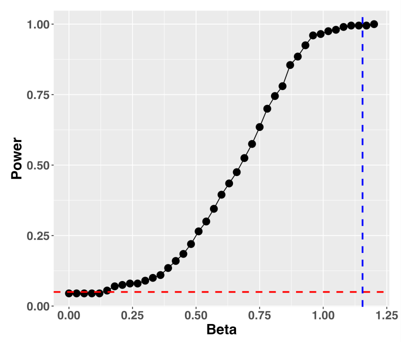
- We are particularly interested in the test's performance when the signal is below the phase transition threshold, i.e., $\beta \in (0, \beta_s]$. For the case $\mathfrak{c}_1 = \mathfrak{c}_2 = \mathfrak{c}_3 = 1/3$, the phase transition threshold is $\beta_s = 2/\sqrt{3}$, as stated in Corollary 3 of Seddik et al. [2024].
- We use the same settings as in Experiment 1, with a significance level of $\alpha = 0.05$. We compute the test's empirical power for different β values with 200 repetitions. All results are summarized in Table 1 and Figures 1 and 2

Experiment 1: test for signal alignment

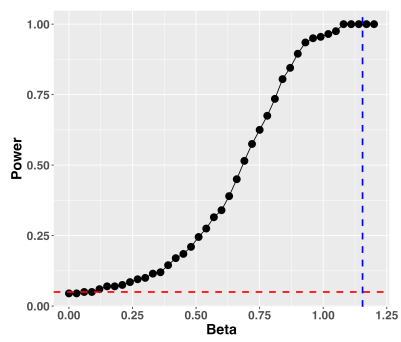
Table 1: Empirical sizes ($\beta = 0$) and powers of $\tilde{\mathcal{T}}_N^{(3)}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(3)})$ under different β's and types of noises $\boldsymbol{X}$ and vectors $\boldsymbol{x}^{(i)}$.

β	$\mathcal{N}(0,1)$, delocalized	$\mathcal{N}(0,1)$, localized	$\mathrm{Unif}(\pm\sqrt{3})$, delocalized	$\mathrm{Unif}(\pm\sqrt{3})$, localized
0	0.045	0.050	0.045	0.055
0.2	0.075	0.055	0.075	0.090
0.4	0.135	0.115	0.145	0.230
0.6	0.345	0.355	0.340	0.685
0.8	0.745	0.730	0.735	0.975
1	0.970	0.980	0.965	1
1.2	1	1	1	1

Experiment 1: test for signal alignment



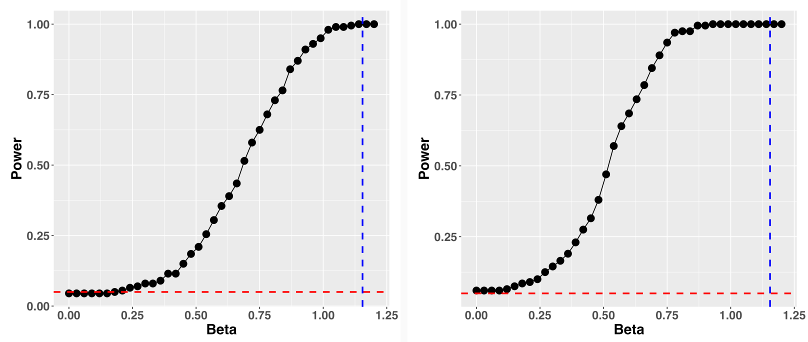
(a) $\boldsymbol{X} \sim \mathcal{N}(0,1)$, delocalized $\boldsymbol{x}^{(i)}$.



(b) $\boldsymbol{X} \sim \mathrm{Unif}(\pm\sqrt{3})$, delocalized $\boldsymbol{x}^{(i)}$.

Figure 1: Power plots of $\tilde{\mathcal{T}}_N^{(3)}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(3)})$ under different β's and types of noises $\boldsymbol{X}$ and delocalized vectors $\boldsymbol{x}^{(i)}$, where the dashed red line is the significance level $\alpha = 0.05$ and the dashed blue line is the threshold of phase transition.

Experiment 1: test for signal alignment



(a) $\boldsymbol{X} \sim \mathcal{N}(0,1)$, localized $\boldsymbol{x}^{(i)}$.

(b) $\boldsymbol{X} \sim \mathrm{Unif}(\pm\sqrt{3})$, localized $\boldsymbol{x}^{(i)}$.

Figure 2: Power plots of $\tilde{\mathcal{T}}_N^{(3)}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(3)})$ under different β's and types of noises $\boldsymbol{X}$ and localized vectors $\boldsymbol{x}^{(i)}$, where the dashed red line is the significance level $\alpha = 0.05$ and the dashed blue line is the threshold of phase transition.

Experiment 2: test for signal matching

- This experiment focuses on the tensor signal matching test. We generate two independent samples, $\boldsymbol{T}^{(0)}$ and $\boldsymbol{T}^{(1)}$, using the following model:

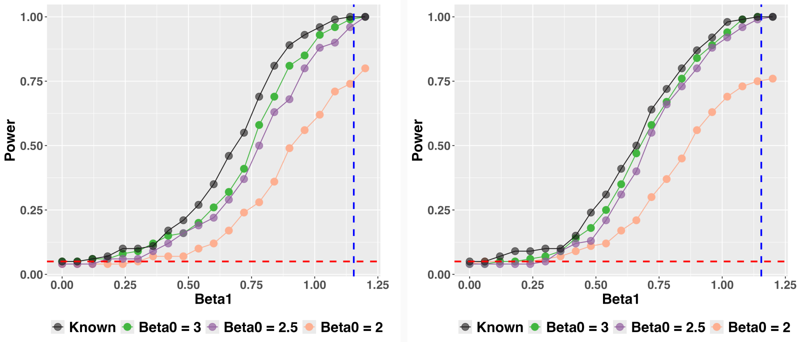
$$\begin{cases} \boldsymbol{T}^{(0)} = \beta_0 \boldsymbol{x}^{(1)} \otimes \boldsymbol{x}^{(2)} \otimes \boldsymbol{x}^{(3)} + \frac{1}{\sqrt{N}} \boldsymbol{X}^{(0)}, \\ \boldsymbol{T}^{(1)} = \beta_1 \boldsymbol{x}^{(1)} \otimes \boldsymbol{x}^{(2)} \otimes \boldsymbol{x}^{(3)} + \frac{1}{\sqrt{N}} \boldsymbol{X}^{(1)}, \end{cases}$$

  where the noise tensors $\boldsymbol{X}^{(0)}$ and $\boldsymbol{X}^{(1)}$ are independent, and the two rank-1 tensor signals are parallel but have different strengths.

- We first apply the tensor unfolding method to estimate the $\hat{\boldsymbol{x}}^{(1)} \otimes \hat{\boldsymbol{x}}^{(2)} \otimes \hat{\boldsymbol{x}}^{(3)}$ using the first tensor data $\boldsymbol{T}^{(0)}$. Then, we test whether $\boldsymbol{T}^{(1)}$ contains a signal along $\hat{\boldsymbol{x}}^{(1)} \otimes \hat{\boldsymbol{x}}^{(2)} \otimes \hat{\boldsymbol{x}}^{(3)}$ or not.

Experiment 2: test for signal matching

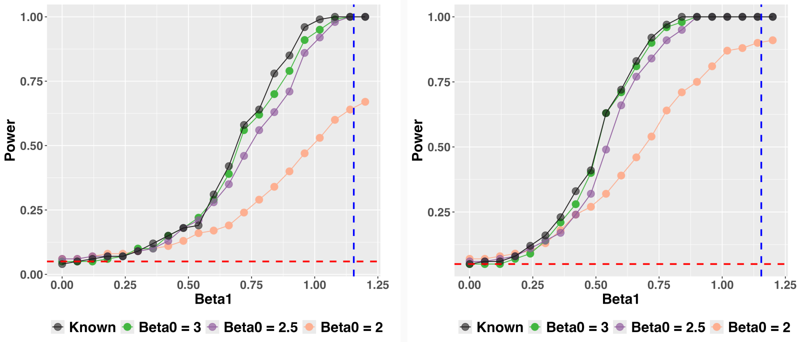
- The main objective of this experiment is to investigate how the values of β_0 and β_1 affect the power of the tensor signal matching test and to compare it with the power of $\tilde{\mathcal{T}}_N^{(3)}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(3)})$ when using known directional vectors.
- We set $\beta_0 = 2, 2.5, 3$ and estimate $\hat{\boldsymbol{x}}^{(1)}, \hat{\boldsymbol{x}}^{(2)}, \hat{\boldsymbol{x}}^{(3)}$ for each β_0. The rest of the setting is essentially the same as in Experiment 2, with the addition of $\beta_1 \in [0, 1.2]$. We compute the empirical power of $\tilde{\mathcal{T}}_N^{(3)}(\hat{\boldsymbol{x}}^{(1)}, \hat{\boldsymbol{x}}^{(2)}, \hat{\boldsymbol{x}}^{(3)})$ and present the power plots in Figures 3 and 4.



(a) $\boldsymbol{X} \sim \mathcal{N}(0,1)$, delocalized $\boldsymbol{x}^{(i)}$.

(b) $\boldsymbol{X} \sim \mathrm{Unif}(\pm\sqrt{3})$, delocalized $\boldsymbol{x}^{(i)}$.

Figure 3: Power plots of $\tilde{T}_N^{(3)}(\hat{\boldsymbol{x}}^{(1)}, \hat{\boldsymbol{x}}^{(2)}, \hat{\boldsymbol{x}}^{(3)})$ under different β_0, β_1 and types of noises $\boldsymbol{X}$ and delocalized vectors $\boldsymbol{x}^{(i)}$. "Known" denotes the empirical power of $\tilde{T}_N^{(3)}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(3)})$, while "Beta0$=a$" represents the empirical power of $\tilde{T}_N^{(3)}(\hat{\boldsymbol{x}}^{(1)}, \hat{\boldsymbol{x}}^{(2)}, \hat{\boldsymbol{x}}^{(3)})$ when $\beta_0 = a$, $a = 2, 2.5, 3$. The dashed red line and blue line indicate the significance level $\alpha = 0.05$ and the threshold of phase transition $\beta_s = 2/\sqrt{3} = 1.1547$, respectively.



(a) $\boldsymbol{X} \sim \mathcal{N}(0,1)$, localized $\boldsymbol{x}^{(i)}$.

(b) $\boldsymbol{X} \sim \mathrm{Unif}(\pm\sqrt{3})$, localized $\boldsymbol{x}^{(i)}$.

Figure 4: Power plots of $\tilde{T}_N^{(3)}(\hat{\boldsymbol{x}}^{(1)}, \hat{\boldsymbol{x}}^{(2)}, \hat{\boldsymbol{x}}^{(3)})$ under different β_0, β_1 and types of noises $\boldsymbol{X}$ and localized vectors $\boldsymbol{x}^{(i)}$. "Known" denotes the empirical power of $\tilde{T}_N^{(3)}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(3)})$, while "Beta0=$a$" represents the empirical power of $\tilde{T}_N^{(3)}(\hat{\boldsymbol{x}}^{(1)}, \hat{\boldsymbol{x}}^{(2)}, \hat{\boldsymbol{x}}^{(3)})$ when $\beta_0 = a$, $a = 2, 2.5, 3$. The dashed red line and blue line indicate the significance level $\alpha = 0.05$ and the threshold of phase transition $\beta_s = 2/\sqrt{3} = 1.1547$, respectively.

References

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Thank you!